

Nordström gravity coupled to point particles in (1 + 1) dimensions

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(Received 26 October 2009; published 18 March 2010)

We consider a (1 + 1)-dimensional model of general relativity that is based on a geometric theory of gravity due to Nordström. We show that the theory is formally equivalent to scalar field theory in flat spacetime, and we exploit this equivalence to solve the initial value problem for the system for the case of point particle sources. We illustrate our results by obtaining several example solutions to the model.

DOI: [10.1103/PhysRevD.81.064022](https://doi.org/10.1103/PhysRevD.81.064022)

PACS numbers: 04.60.Kz, 03.50.-z, 04.20.-q

I. INTRODUCTION

The Einstein field equations do not yield a viable theory of gravity in (1 + 1) dimensions, as can be understood from the following considerations. In (1 + 1) dimensions the Riemann and Ricci tensors can be expressed as

$$\begin{aligned} R^\gamma_{\nu\alpha\beta} &= (R/2)g^{\gamma\mu}(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}), \\ R_{\nu\beta} &= R^\gamma_{\nu\gamma\beta} = (R/2)g_{\nu\beta}, \end{aligned} \quad (1)$$

where $g_{\mu\nu}$ is the metric tensor and $R = g^{\nu\beta}R_{\nu\beta}$ is the curvature scalar [1]. Equation (1) implies that the Einstein tensor $G_{\mu\nu}$ vanishes identically,

$$G_{\mu\nu} = R_{\mu\nu} - (1/2)Rg_{\mu\nu} = 0, \quad (2)$$

so the Einstein field equations reduce to

$$G_{\mu\nu} = 8\pi GT_{\mu\nu} = 0, \quad (3)$$

where G is the gravitational constant and $T_{\mu\nu}$ is the energy-momentum tensor. Thus, the Einstein field equations do not constrain the metric tensor and simply state that the energy-momentum tensor vanishes.

It is possible, however, to construct a (1 + 1)-dimensional analog to general relativity by replacing Einstein's field equations with the field equation

$$R = 4Gg_{\alpha\beta}T^{\alpha\beta}. \quad (4)$$

This theory was proposed as a model of general relativity in the 1980s, and has been investigated by a number of authors [2–5]. Features of the model that have been studied include black hole solutions [6], gravitational collapse [7], and chaotic particle dynamics [8,9]. Quantum aspects of the theory have also been investigated [10]. We can view the theory as the direct (1 + 1)-dimensional analog of a generally covariant theory of gravity in (3 + 1) dimensions that was proposed by Nordström [11,12]. The Nordström field equations are

$$R = 24\pi Gg_{\alpha\beta}T^{\alpha\beta}, \quad C_{\alpha\beta\mu\nu} = 0, \quad (5)$$

where $C_{\alpha\beta\mu\nu}$ is the Weyl tensor corresponding to $g_{\mu\nu}$. In (1 + 1) dimensions the Weyl tensor vanishes identically, so the Nordström field equations are equivalent to Eq. (4) up to a trivial rescaling of the gravitational constant.

In this paper we consider the special case of (1 + 1)-dimensional Nordström gravity coupled to point particle sources. We show that the theory is formally equivalent to scalar field theory in flat spacetime, and we use this equivalence to solve the initial value problem for the system. The dynamics of systems of point particles is of considerable interest in general relativity [13–16], and has also been studied in the context of (1 + 1)-dimensional Nordström gravity [4]. In (1 + 1)-dimensional Nordström gravity there is no gravitational radiation, but the gravitational interaction of the particles still gives rise to non-trivial particle dynamics. By exploiting the formal equivalence of Nordström gravity and scalar field theory in flat spacetime, we provide a new way of thinking about this problem.

Our solution to the initial value problem relies on a calculation of the retarded fields and radiation reaction force for a point particle coupled to a classical scalar field in (1 + 1) dimensions. As a side result, we show that the radiation reaction force on the particle vanishes, despite the fact that the particle can scatter and emit radiation. We give a physical justification for this result by using the equivalence between scalar field theory and Nordström gravity. The result is of interest because it shows how issues in the foundations of classical electrodynamics, such as the radiation reaction force on a point particle, carry over to the case of a scalar field in (1 + 1) dimensions [17,18].

The paper is organized as follows. In Sec. II we describe Nordström gravity coupled to point particle sources in (1 + 1) dimensions. In Sec. III we show that this theory is formally equivalent to scalar field theory in flat spacetime, and we derive the equations of motion for the system under the flat spacetime interpretation. In Sec. IV we calculate the retarded fields generated by a point particle coupled to a scalar field. In Sec. V we use our expressions for the retarded fields to reformulate the flat spacetime equations of motion in terms of new dynamical degrees of freedom for the field. In Sec. VI we use the new equations of motion to formally solve the initial value problem for the system. In Sec. VII we present several example solutions.

The following notation is used in this paper. The function $\epsilon(x)$ is the sign function, defined such that $\epsilon(x) = 1$ if $x > 0$, $\epsilon(x) = 0$ if $x = 0$, and $\epsilon(x) = -1$ if $x < 0$.

The function $\theta(x)$ is the step function, defined such that $\theta(x) = 1$ if $x > 0$, $\theta(x) = 1/2$ if $x = 0$, and $\theta(x) = 0$ if $x < 0$. The Minkowski metric tensor $\eta_{\mu\nu}$ is defined such that $\eta_{00} = 1$, $\eta_{11} = -1$, $\eta_{01} = \eta_{10} = 0$. The Levi-Civita tensor $\epsilon_{\mu\nu}$ is defined such that $\epsilon_{01} = -\epsilon_{10} = 1$, $\epsilon_{00} = \epsilon_{11} = 0$. We define $a \cdot b \equiv a_\mu b^\mu$ for vectors a^μ and b^μ .

II. NORDSTRÖM GRAVITY COUPLED TO POINT PARTICLES

Consider a point particle of mass m moving along a trajectory $z^\mu(s)$, where s is the proper time defined relative to the metric tensor $g_{\mu\nu}$. One can show that the energy-momentum tensor for the particle is

$$T^{\alpha\beta}(x) = mg^{-1/2} \int u^\alpha u^\beta \delta^{(2)}(x - z(s)) ds, \quad (6)$$

where $g \equiv -\det g_{\mu\nu}$ and $u^\mu(s) \equiv dz^\mu(s)/ds$ is the particle velocity. Substituting Eq. (6) into the field equation (4), we find that

$$R = 4Gmg^{-1/2} \int \delta^{(2)}(x - z(s)) ds. \quad (7)$$

In Nordström gravity, as in general relativity, freely falling particles move along geodesics, so the particle equation of motion is given by the geodesic equation:

$$\frac{du^\mu}{ds} + \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta = 0. \quad (8)$$

We have described the equations of motion for the special case of a single point particle, but it is straightforward to generalize to the case of multiple point particles.

An important property of Nordström gravity in $(1+1)$ dimensions is that the gravitational field is not dynamical; that is, there are no gravitational waves. This property follows directly from the field equation (4): note that if the energy-momentum tensor vanishes then the curvature scalar vanishes, and hence, by Eq. (1), the Riemann tensor vanishes. So in vacuum there is no curvature, and hence no physical gravitational waves. One consequence of this fact, which will be important for later considerations, is that a system of gravitationally bound particles cannot dissipate energy by emitting gravitational radiation. As we shall see, although the gravitational field does not support freely propagating radiation, it does mediate forces among the particles and gives rise to nontrivial particle dynamics. In this respect Nordström gravity in $(1+1)$ dimensions is analogous to Newtonian gravity.

III. FLAT SPACETIME INTERPRETATION

We will now show that Nordström gravity can be reinterpreted as a scalar field theory in flat spacetime. In $(1+1)$ dimensions spacetime is conformally flat for all metrics; that is, one can always choose a coordinate system (t, x) such that the metric tensor takes the form $g_{\mu\nu} = e^{2\phi} \eta_{\mu\nu}$,

where ϕ is a scalar potential and $\eta_{\mu\nu}$ is the Minkowski metric tensor [19]. By working in a coordinate system in which the metric takes this form, we will show that the field equation (7) and the particle equation of motion (8) can be reinterpreted as describing a point particle coupled to a scalar field in flat spacetime. In order to ensure that the metric is sufficiently smooth that the coordinate transformation can be performed, one can consider a spatially extended particle and then take the limit as the size of the particle goes to zero (spatially extended particles coupled to fields are discussed in Ref. [17]). In what follows it will be useful to note that the Christoffel symbols corresponding to $g_{\mu\nu}$ are

$$\Gamma^\mu_{\alpha\beta} = \delta^\mu_\beta \partial_\alpha \phi + \delta^\mu_\alpha \partial_\beta \phi - \eta_{\alpha\beta} \eta^{\mu\nu} \partial_\nu \phi, \quad (9)$$

and the curvature scalar is

$$R = -2e^{-2\phi} \square \phi. \quad (10)$$

First we will reinterpret the particle equation of motion (8). If we view the particle as moving in a flat spacetime described by the metric tensor $\eta_{\mu\nu}$, then the proper time τ and the particle velocity w^μ are given by

$$d\tau = (\eta_{\mu\nu} dz^\mu dz^\nu)^{1/2} = (\eta_{\mu\nu} u^\mu u^\nu)^{1/2} ds = e^{-\phi} ds, \\ w^\mu = \frac{dz^\mu}{d\tau} = \frac{ds}{d\tau} \frac{dz^\mu}{ds} = e^\phi u^\mu. \quad (11)$$

We can use Eqs. (9) and (11) to rewrite the particle equation of motion (8) in terms of the flat spacetime quantities τ , w^μ , and ϕ :

$$\frac{dw^\mu}{d\tau} + F_\nu(z)(w^\mu w^\nu - \eta^{\mu\nu}) = 0. \quad (12)$$

Here $F_\mu \equiv \partial_\mu \phi$ is a vector whose components describe the fields corresponding to the potential ϕ ; we will denote these fields by $B \equiv F_0 = \partial_t \phi$ and $E \equiv F_1 = \partial_x \phi$. Equation (12) is the particle equation of motion in the flat spacetime interpretation.

Next we will reinterpret the field equation (7). Using Eqs. (6) and (11), we find that the trace of the energy-momentum tensor is given by

$$g_{\alpha\beta} T^{\alpha\beta} = e^{-\phi} \rho, \quad (13)$$

where

$$\rho(x) = m \int \delta^{(2)}(x - z(\tau)) d\tau \quad (14)$$

is the mass density as defined in the flat spacetime interpretation. We can use Eqs. (10) and (13) to rewrite the field equation (4) in terms of ϕ and ρ :

$$\square \phi = \partial_\mu F^\mu = -2Ge^\phi \rho. \quad (15)$$

Equation (15) is the field equation for ϕ in the flat spacetime interpretation.

In summary, we have shown that from one point of view the spacetime is flat and the particle is coupled to a scalar potential ϕ . The particle equation of motion is given by Eq. (12) and the field equation for the potential ϕ is given by Eq. (15). From another point of view the spacetime is curved, with metric tensor $g_{\mu\nu} = e^{2\phi} \eta_{\mu\nu}$, and the particle falls freely. The particle equation of motion is given by the geodesic equation (8) and the field equation for the metric tensor is given by Eq. (7).

Note that the particle equation of motion (12) and the field equation (15) can be derived from the Lagrangian

$$\mathcal{L} = (1/4G)(\partial_\mu \phi)(\partial^\mu \phi) - e^\phi \rho. \quad (16)$$

In terms of the E and B fields, the particle equation of motion (12) and the field equation (15) become

$$\begin{aligned} \frac{d}{dt}(\beta\gamma) &= -\gamma(E(z) + \beta B(z)), \\ \partial_t B - \partial_x E &= -2Ge^\phi \rho, \end{aligned} \quad (17)$$

where $\gamma \equiv w_0$ and $\beta \equiv w^1/w_0$.

In the flat spacetime interpretation, the scalar field is dynamical and supports freely propagating waves, but, as we remarked in the previous section, in the curved spacetime interpretation the gravitational field is not dynamical and there are no physical gravitational waves. Thus, freely propagating waves in the flat spacetime interpretation correspond to gauge waves in the curved spacetime interpretation. We can demonstrate this correspondence by explicitly writing down a coordinate transformation that eliminates the waves. For freely propagating waves in the flat spacetime interpretation, we can express the potential ϕ in the form

$$\phi(t, x) = f_+(t + x) + f_-(t - x), \quad (18)$$

where f_+ and f_- describe left-moving and right-moving waves. Define coordinates (w_+, w_-) , (\bar{w}_+, \bar{w}_-) , and (\bar{t}, \bar{x}) such that

$$w_\pm = t \pm x, \quad \bar{w}_\pm = \int_0^{w_\pm} e^{2f_\pm(u)} du = \bar{t} \pm \bar{x}. \quad (19)$$

We can then express the metric for the curved spacetime interpretation as

$$\begin{aligned} ds^2 &= e^{2\phi(t,x)}(dt^2 - dx^2) = e^{2f_+(w_+) + 2f_-(w_-)} dw_+ dw_- \\ &= d\bar{w}_+ d\bar{w}_- = d\bar{t}^2 - d\bar{x}^2. \end{aligned} \quad (20)$$

Thus, in the (\bar{t}, \bar{x}) coordinate system the metric reduces to the Minkowski metric. Note that the potential ϕ is pure gauge only for the case of freely propagating waves; in the presence of matter, there is a component to ϕ that cannot be eliminated by a coordinate transformation. This component gives rise to an attractive force between pairs of point particles.

We can decompose each solution to the field equation (15) into the sum $\phi = \phi_r + \phi_{\text{in}}$ of an inhomogeneous

solution ϕ_r , which describes the retarded potential generated by the particle, and a homogeneous solution ϕ_{in} , which describes the remainder of the potential. We can further decompose the homogeneous solution into the sum $\phi_{\text{in}} = \phi_0 + \phi_b + \phi_{\text{rad}}$ of a constant potential ϕ_0 , a potential $\phi_b(x) = F_b^\mu x_\mu$ that describes constant background fields $B_b \equiv F_b^0$ and $E_b \equiv -F_b^1$, and a potential $\phi_{\text{rad}}(x)$ that describes freely propagating incoming radiation. By definition, the potentials ϕ_r , ϕ_{in} , and ϕ_{rad} satisfy the field equations

$$\square \phi_r = -2Ge^\phi \rho, \quad \square \phi_{\text{in}} = \square \phi_{\text{rad}} = 0. \quad (21)$$

Given initial data $\phi(0, x)$, $B(0, x)$, $z^\mu(0)$, $w^\mu(0)$, we would like to solve for the time evolution of the system. As we will show in Sec. VI, we can formally solve this initial value problem by reformulating the equations of motion (12) and (15) in such a way that the dynamical variables for the field are taken to be $E_{\text{in}} \equiv \partial_x \phi_{\text{in}}$ and $B_{\text{in}} \equiv \partial_t \phi_{\text{in}}$ rather than ϕ and B . We will show how to reformulate the equations of motion in Sec. V, but before doing so we first need to calculate the retarded fields generated by the particle.

IV. RETARDED FIELDS

We can calculate the retarded fields generated by the particle as follows. We substitute Eq. (14) for the mass density into the field equation (21) for the retarded potential to obtain

$$\square \phi_r(x) = -2a \int f(\tau) \delta^{(2)}(x - z(\tau)) d\tau, \quad (22)$$

where $a \equiv Gm$ and

$$f(\tau) \equiv e^{\phi(z(\tau))}. \quad (23)$$

We will calculate the retarded potential $\phi_r(x)$ by solving Eq. (22) for an arbitrary function $f(\tau)$ and then afterward self-consistently imposing Eq. (23). For an arbitrary function $f(\tau)$, we can express the solution to Eq. (22) as

$$\phi_r(x) = -a \int f(\tau) G_r(x - z(\tau)) d\tau, \quad (24)$$

where $G_r(x)$ is the retarded Green function for the inhomogeneous wave equation:

$$\begin{aligned} G_r(x) &= \theta(x^0 - |x^1|) = \theta(x^0) \theta(x \cdot x), \\ \square G_r(x) &= 2\delta^{(2)}(x). \end{aligned} \quad (25)$$

The retarded fields $F_r^\mu(x)$ corresponding to the potential $\phi_r(x)$ are given by

$$F_r^\mu(x) = \partial^\mu \phi_r(x) = -a \int f(\tau) \partial^\mu G_r(r) d\tau, \quad (26)$$

where $r^\mu(x, \tau) \equiv x^\mu - z^\mu(\tau)$. We will evaluate Eq. (26) by considering as separate cases events x^μ that lie on the

particle world line and events x^μ that do not lie on the particle world line.

Let us first evaluate Eq. (26) for events x^μ that lie on the particle world line. We can evaluate $\partial^\mu G_r(r)$ as follows. By symmetry $\partial^\mu G_r(r) = g(r)r^\mu$ for some function $g(r)$. Thus, we can express the derivative of $G_r(r)$ with respect to proper time as

$$\begin{aligned} \frac{d}{d\tau} G_r(r) &= \frac{dr^\mu}{d\tau} \partial_\mu G_r(r) = -w^\mu \partial_\mu G_r(r) \\ &= -(r \cdot w)g(r). \end{aligned} \quad (27)$$

Thus,

$$\partial_\mu G_r(r) = g(r)r_\mu = -(r \cdot w)^{-1}r_\mu \frac{d}{d\tau} G_r(r). \quad (28)$$

Substituting Eq. (28) into Eq. (26), we find that

$$F_r^\mu(x) = a \int f(\tau)(r \cdot w)^{-1}r^\mu \frac{d}{d\tau} G_r(r) d\tau. \quad (29)$$

Since the event x^μ lies on the particle world line, there is some proper time τ' such that $x^\mu = z^\mu(\tau')$. It follows that $r^\mu = z^\mu(\tau') - z^\mu(\tau)$ and $G_r(r) = \theta(\tau' - \tau)$, so

$$F_r^\mu(z(\tau')) = -a \int f(\tau)(r \cdot w)^{-1}r^\mu \delta(\tau' - \tau) d\tau. \quad (30)$$

Let us define $\lambda = \tau' - \tau$ and expand r^μ in λ . We find that

$$\begin{aligned} r^\mu &= z^\mu(\tau') - z^\mu(\tau) = \lambda w^\mu(\tau) + \mathcal{O}(\lambda^2), \\ (r \cdot w)^{-1}r^\mu &= w^\mu(\tau) + \mathcal{O}(\lambda). \end{aligned} \quad (31)$$

Thus, if we perform the integral over τ in Eq. (30) and then replace τ' with τ , we find that

$$F_r^\mu(z(\tau)) = -af(\tau)w^\mu(\tau). \quad (32)$$

Now let us evaluate Eq. (26) for events x^μ that do not lie on the particle world line. For such events we can define retarded and advanced proper times $\tau_r(x)$ and $\tau_a(x)$ by

$$\begin{aligned} r(x, \tau_r) \cdot r(x, \tau_r) &= r(x, \tau_a) \cdot r(x, \tau_a) = 0, \\ z^0(\tau_r) &< x^0 < z^0(\tau_a). \end{aligned} \quad (33)$$

Note that

$$\partial^\mu G_r(r) = 2r^\mu \theta(r^0) \delta(r \cdot r) = (r \cdot w)^{-1}r^\mu \delta(\tau - \tau_r), \quad (34)$$

where we have used the fact that

$$\delta(r \cdot r) = (1/2)(r \cdot w)^{-1}(\delta(\tau - \tau_r) - \delta(\tau - \tau_a)). \quad (35)$$

Substituting Eq. (34) into Eq. (26) and performing the integral on τ , we find that

$$F_r^\mu(x) = -af(\tau_r)(r(x, \tau_r) \cdot w(\tau_r))^{-1}r^\mu(x, \tau_r), \quad (36)$$

where $\tau_r(x)$ is the retarded proper time for the event x^μ .

For the remainder of this section we will assume that r^μ and w^μ are evaluated at the retarded proper time $\tau_r(x)$ for

the event x^μ ; that is, we will view r^μ and w^μ as vector fields defined such that $r^\mu(x) \equiv r^\mu(x, \tau_r(x))$ and $w^\mu(x) \equiv w^\mu(\tau_r(x))$. Let us define a vector field $n^\mu(x)$ by

$$n^\mu \equiv (r \cdot w)^{-1}r^\mu - w^\mu. \quad (37)$$

From the fact that r^μ is lightlike ($r \cdot r = 0$), it follows that w^μ and n^μ are orthonormal:

$$w \cdot w = -n \cdot n = 1, \quad w \cdot n = 0. \quad (38)$$

Note that n^μ is spacelike. Using the orthonormality relations given in Eq. (38), one can show that

$$n_\mu(x) = \epsilon(x^1 - z^1(\tau_r))\epsilon_{\mu\nu}w^\nu(\tau_r). \quad (39)$$

From Eqs. (37) and (39), it follows that

$$\begin{aligned} r^\mu &= (r \cdot w)(w^\mu + n^\mu) \\ &= (r \cdot w)(\eta^{\mu\nu} + \epsilon(x^1 - z^1(\tau_r))\epsilon^{\mu\nu})w_\nu(\tau_r). \end{aligned} \quad (40)$$

Note that $r \cdot w$ can be interpreted as the distance from the particle to the event x^μ , as measured in the instantaneous rest frame of the particle at the retarded proper time $\tau_r(x)$. A diagram of the vectors r^μ , w^μ , and n^μ for a specific event x^μ is shown in Fig. 1.

If we substitute Eq. (40) for r^μ into Eq. (36) for the retarded fields, we find that

$$\begin{aligned} F_r^\mu(x) &= -af(\tau_r)(w^\mu(\tau_r) + n^\mu(x)) \\ &= -af(\tau_r)(\eta^{\mu\nu} + \epsilon(x^1 - z^1(\tau_r))\epsilon^{\mu\nu})w_\nu(\tau_r). \end{aligned} \quad (41)$$

For events x^μ to the right of the particle ($x^1 > z^1(\tau_r)$), the retarded E and B fields are

$$B_r(x) = -E_r(x) = -af(\tau_r)\gamma(\tau_r)(1 + \beta(\tau_r)). \quad (42)$$

For events x^μ to the left of the particle ($x^1 < z^1(\tau_r)$), the

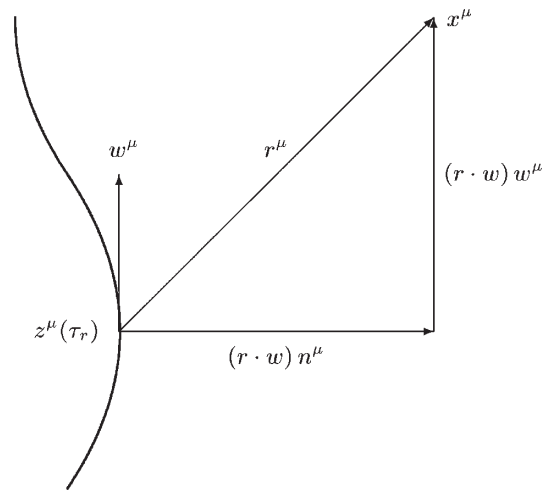


FIG. 1. Diagram of vectors r^μ , w^μ , and n^μ , drawn in the instantaneous rest frame of the particle at the retarded proper time $\tau_r(x)$ of event x^μ . The curved line is the particle world line.

retarded E and B fields are

$$B_r(x) = E_r(x) = -af(\tau_r)\gamma(\tau_r)(1 - \beta(\tau_r)). \quad (43)$$

V. RETARDED FORMULATION

We will now use the results of the previous section to reformulate the equations of motion (12) and (15) in such a way that the dynamical variables for the field are taken to be E_{in} and B_{in} rather than ϕ and B .

First we will determine the new particle equation of motion. Using the decomposition $\phi = \phi_r + \phi_{\text{in}}$, we can express the particle equation of motion (12) as

$$\frac{dw_\mu}{d\tau} = -F_r^\nu(z)(w_\mu w_\nu - \eta_{\mu\nu}) - F_{\text{in}}^\nu(z)(w_\mu w_\nu - \eta_{\mu\nu}), \quad (44)$$

where $F_{\text{in}}^\mu \equiv \partial^\mu \phi_{\text{in}}$. The second term on the right-hand side of Eq. (44) describes the force exerted on the particle by the background fields and by incoming radiation, and the first term describes the radiation reaction force exerted on the particle by its own retarded fields. If we substitute for $F_r^\nu(z)$ using Eq. (32), we find that the radiation reaction force vanishes and Eq. (44) reduces to

$$\frac{dw_\mu}{d\tau} = -F_{\text{in}}^\nu(z)(w_\mu w_\nu - \eta_{\mu\nu}). \quad (45)$$

The fact that the radiation reaction force vanishes can be physically understood from the following considerations. In Sec. II we observed that in the curved spacetime interpretation there is no gravitational radiation, so a system of gravitationally bound particles cannot dissipate energy. This observation implies that in the flat spacetime interpretation there can be no radiation damping and the radiation reaction force must vanish.

Next we will obtain an equation of motion for $f(\tau)$ by self-consistently imposing Eq. (23). We first differentiate Eq. (23) with respect to τ to obtain

$$\frac{df}{d\tau} = w_\mu F^\mu(z)f. \quad (46)$$

If we substitute for $F^\mu(z)$ using the decomposition $F^\mu(z) = F_r^\mu(z) + F_{\text{in}}^\mu(z)$ and for $F_r^\mu(z)$ using Eq. (32), we obtain the equation of motion

$$\frac{df}{d\tau} = -af^2 + w_\mu F_{\text{in}}^\mu(z)f. \quad (47)$$

Finally we will determine the field equations for the in fields. From the field equation for ϕ_{in} given in Eq. (21) and the definition $F_{\text{in}}^\mu = \partial^\mu \phi_{\text{in}}$, it follows that

$$\partial_\mu F_{\text{in}}^\mu = 0, \quad \epsilon_{\mu\nu} \partial^\mu F_{\text{in}}^\nu = 0. \quad (48)$$

Equations (45), (47), and (48) are the equations of motion for the system in the retarded formulation. In terms of the E and B fields these equations of motion become

$$\begin{aligned} \frac{d}{dt}(\beta\gamma) &= -\gamma(E_{\text{in}}(z) + \beta B_{\text{in}}(z)), \\ \frac{df}{dt} &= -(a/\gamma)f^2 + (B_{\text{in}}(z) + \beta E_{\text{in}}(z))f, \\ \partial_t E_{\text{in}} &= \partial_x B_{\text{in}}, \quad \partial_t B_{\text{in}} = \partial_x E_{\text{in}}. \end{aligned} \quad (49)$$

So far we have considered the case of a single particle, but it is straightforward to generalize to the case of N particles. Let m_i denote the mass of particle number i , let z_i^μ and w_i^μ denote its position and velocity, and let $F_{r,i}^\mu$ denote its retarded fields. We define $a_i \equiv Gm_i$ and $f_i \equiv e^{\phi(z_i)}$. The equations of motion for w_i^μ and f_i are given by

$$\begin{aligned} \frac{d}{d\tau} w_{i,\mu} &= -(F_{\text{in}}^\nu(z_i) + \sum_{j \neq i} F_{r,j}^\nu(z_i))(w_{i,\mu} w_{i,\nu} - \eta_{\mu\nu}), \\ \frac{d}{d\tau} f_i &= -a_i f_i^2 + w_{i,\mu} \left(F_{\text{in}}^\mu(z_i) + \sum_{j \neq i} F_{r,j}^\mu(z_i) \right), \end{aligned} \quad (50)$$

and the equations of motion for F_{in}^μ are given by Eq. (48).

VI. SOLUTION TO THE INITIAL VALUE PROBLEM

We can use the retarded formulation of the equations of motion to formally solve the initial value problem for the coupled particle-field system. Suppose we are given initial data $\phi(0, x)$, $B(0, x)$, $z^\mu(0)$, and $w^\mu(0)$. We define $f(0) \equiv e^{\phi(z(0))}$ and $E(0, x) \equiv \partial_x \phi(0, x)$. For $\tau < 0$ we define

$$f(\tau) \equiv f(0), \quad z^\mu(\tau) \equiv z^\mu(0) + w^\mu(0)\tau, \quad w^\mu(\tau) \equiv w^\mu(0). \quad (51)$$

Using the definitions given in Eq. (51) together with Eq. (41) for the retarded fields, we can define retarded fields $F_r^\mu(0, x)$ at time $t = 0$. We then define in fields at time $t = 0$ by

$$F_{\text{in}}^\mu(0, x) \equiv F^\mu(0, x) - F_r^\mu(0, x). \quad (52)$$

We are now ready to solve for the time evolution of the system. First we obtain the in fields at all times using d'Alembert's solution to the wave equation:

$$\begin{aligned} E_{\text{in}}(t, x) &= (1/2)(E_{\text{in}}(0, x - t) + E_{\text{in}}(0, x + t) \\ &\quad + B_{\text{in}}(0, x + t) - B_{\text{in}}(0, x - t)), \end{aligned} \quad (53)$$

$$\begin{aligned} B_{\text{in}}(t, x) &= (1/2)(B_{\text{in}}(0, x - t) + B_{\text{in}}(0, x + t) \\ &\quad + E_{\text{in}}(0, x + t) - E_{\text{in}}(0, x - t)). \end{aligned} \quad (54)$$

Using Eqs. (53) and (54) for the in fields, we determine the particle trajectory $z^\mu(\tau)$ by integrating the particle equation of motion (45) subject to the initial conditions $z^\mu(0)$ and $w^\mu(0)$. Using the in fields and the particle trajectory, we determine $f(\tau)$ by integrating the self-consistency equation (47) subject to the initial condition $f(0)$. Finally, using the particle trajectory and $f(\tau)$, we calculate the

retarded fields using Eq. (41). We have now completely solved for the time evolution of the system.

VII. EXAMPLE SOLUTIONS

We will now consider several example solutions. For these solutions we will assume that there is no incoming radiation ($\phi_{\text{rad}} = 0$). We will work in a specific reference frame (t, x) and assume that the background E -field relative to this frame vanishes ($E_b = 0$), but we will allow for a nonzero background B -field B_b .

For our first example we take the particle to start out at $t = 0$ at rest at the origin. According to the equation of motion (45), the particle remains at rest. Thus $\tau = t$ and the self-consistency equation (47) reduces to

$$\frac{df}{dt} = -af^2 + B_b f. \quad (55)$$

It is straightforward to integrate Eq. (55) subject to the initial condition $f(0) = f_0$. For $f_0 = 0$ we find that $f(t) = 0$. For $f_0 > 0$ and $B_b \neq 0$ we find that

$$f(t)/f_0 = (B_b/af_0)[1 + ((B_b/af_0) - 1)e^{-B_b t}]^{-1}, \quad (56)$$

and for $f_0 > 0$ and $B_b = 0$ we find that

$$f(t)/f_0 = (1 + af_0 t)^{-1}. \quad (57)$$

In Fig. 2(a) we plot these solutions for several values of the background field B_b .

The retarded proper time for the event (t, x) is $\tau_r(t, x) = t - |x|$, so from Eqs. (42) and (43) it follows that the E and B fields are given by

$$\begin{aligned} B(t, x) &= B_b + B_r(t, x) = B_b - af(t - |x|), \\ E(t, x) &= E_r(t, x) = af(t - |x|)\epsilon(x), \end{aligned} \quad (58)$$

where $f(t) \equiv f_0$ for $t < 0$. The initial state of the system is

$$\begin{aligned} f(0) &= f_0, & B(0, x) &= B_b - af_0, \\ E(0, x) &= af_0\epsilon(x). \end{aligned} \quad (59)$$

For $B_b > 0$, as $t \rightarrow \infty$ we find that

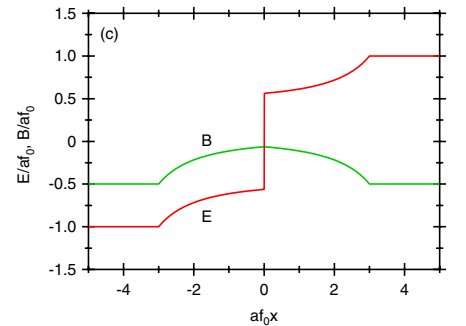
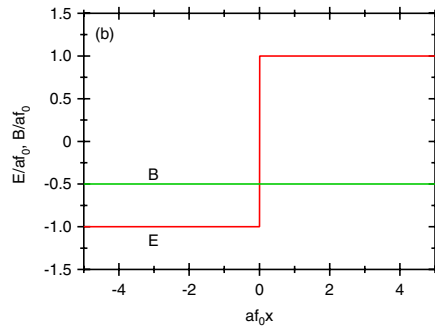
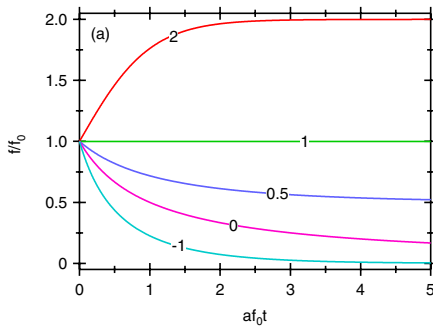


FIG. 2 (color online). Stationary particle. (a) f/f_0 versus $af_0 t$ for $B_b/af_0 = -1, 0, 0.5, 1, 2$. (b, c) E/af_0 and B/af_0 versus $af_0 x$ for $B_b/af_0 = 0.5$ at times $af_0 t = 0, 3$.

$$f(t) \rightarrow B_b/a, \quad B(t, x) \rightarrow 0, \quad E(t, x) \rightarrow B_b\epsilon(x). \quad (60)$$

For $B_b \leq 0$, as $t \rightarrow \infty$ we find that

$$f(t) \rightarrow 0, \quad B(t, x) \rightarrow B_b, \quad E(t, x) \rightarrow 0. \quad (61)$$

Note that we only obtain a static solution if $B_b = af_0$, which implies that $B = 0$, or if $f_0 = 0$, which implies that $E = 0$. A configuration in which both E and B are nonzero cannot be static, since for such a configuration there is a nonzero energy flux $T_f^{01} = -(1/2G)EB$, where

$$T_f^{\alpha\beta} \equiv (2G)^{-1}(F^\alpha F^\beta - (1/2)\eta^{\alpha\beta}F \cdot F) \quad (62)$$

is the energy-momentum tensor for the field. In Figs. 2(b) and 2(c) we plot the E and B fields at times $t/af_0 = 0, 3$ for the case $B_b/af_0 = 0.5$.

For our second example we take the particle to start out at $t = 0$ at the origin with a finite velocity. We take the background B -field to be $B_b = af_0$, so at $t = 0$ the E and B fields are given by the static solution

$$B(0, x) = 0, \quad E(0, x) = af_0\epsilon(x). \quad (63)$$

The initial state for the particle is given by $z^1(0) = 0$, $u(0) = u_0$, $f(0) = f_0$, where $u \equiv w^1 = \beta\gamma$ is the proper velocity. This choice of initial state describes a situation in which the particle is at rest at the origin for $t < 0$ and is given an impulsive momentum kick at $t = 0$. The particle equation of motion (45) and self-consistency equation (47) become

$$\frac{du}{dt} = -B_b u = -af_0 u, \quad \frac{df}{dt} = a(f_0 - f/\gamma)f. \quad (64)$$

We can integrate these equations of motion subject to the initial conditions. We find that

$$\begin{aligned} u(t)/u_0 &= e^{-af_0 t}, \\ f(t)/f_0 &= e^{af_0 t}[1 + (e^{2af_0 t} + u_0^2)^{1/2} - (1 + u_0^2)^{1/2}]^{-1}. \end{aligned} \quad (65)$$

In Fig. 3(a) we plot f/f_0 for several values of u_0 . From

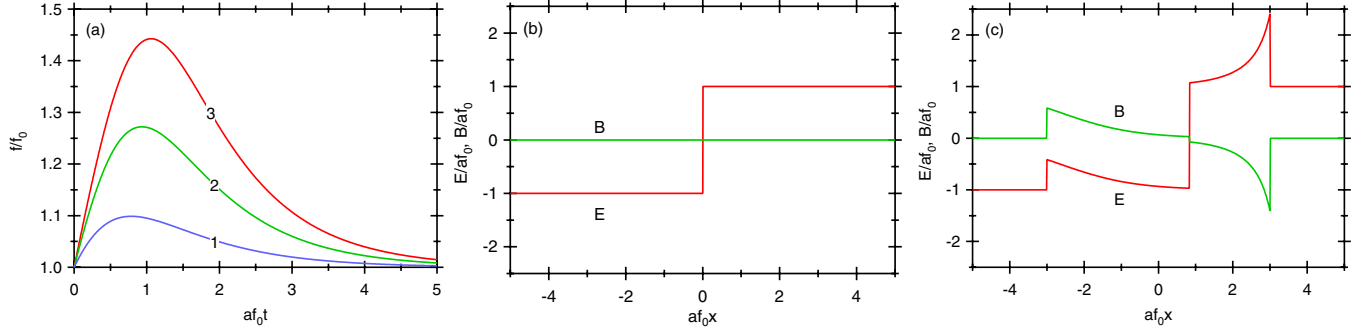


FIG. 3 (color online). Moving particle. (a) f/f_0 versus af_0t for $u_0 = 1, 2, 3$. (b, c) E/af_0 and B/af_0 versus af_0x for $u_0 = 1$ at times $af_0t = 0, 3$.

Eqs. (65) we see that the proper velocity u is exponentially damped. The damping of the particle velocity may appear to violate the Lorentz invariance of the theory, but this is not the case. The background B -field B_b picks out a preferred reference frame and exerts a damping force on the particle that causes its velocity relative to this frame to decay with time.

We can use the expressions for u and f given in Eqs. (65) together with the expressions for the retarded fields given in Eqs. (42) and (43) to calculate the E and B fields. In Figs. 3(b) and 3(c) we plot the E and B fields at times $t/af_0 = 0, 3$ for the case $u_0 = 1$. Note the pulses of radiation that propagate outward to the left and right.

For our third example we consider the case of two particles of equal mass. We take the particles to start out

at $t = 0$ at the origin with equal and opposite velocities. By symmetry,

$$\begin{aligned} z_1^0 = z_2^0 \equiv z^0, \quad z_1^1 = -z_2^1 \equiv z^1, \quad w_1^0 = w_2^0 \equiv w^0, \\ w_1^1 = -w_2^1 \equiv w^1, \quad f_1 = f_2 \equiv f. \end{aligned} \quad (66)$$

We will take the background B -field to be $B_b = 2af_0$, so the initial E and B fields are given by the static solution

$$B(0, x) = 0, \quad E(0, x) = 2af_0\epsilon(x). \quad (67)$$

Because of the symmetry conditions given in Eq. (66), the equations of motion (50) reduce to

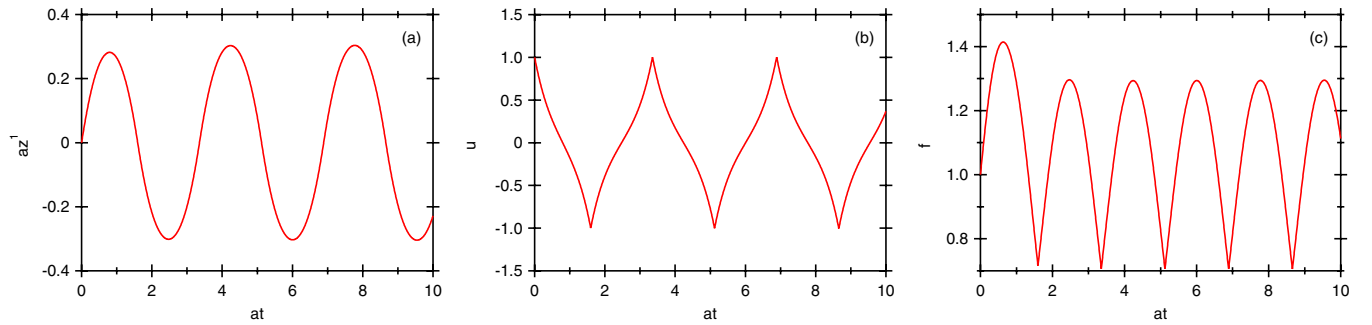


FIG. 4 (color online). Two particles. (a) az^1 versus at . (b) u versus at . (c) f versus at .

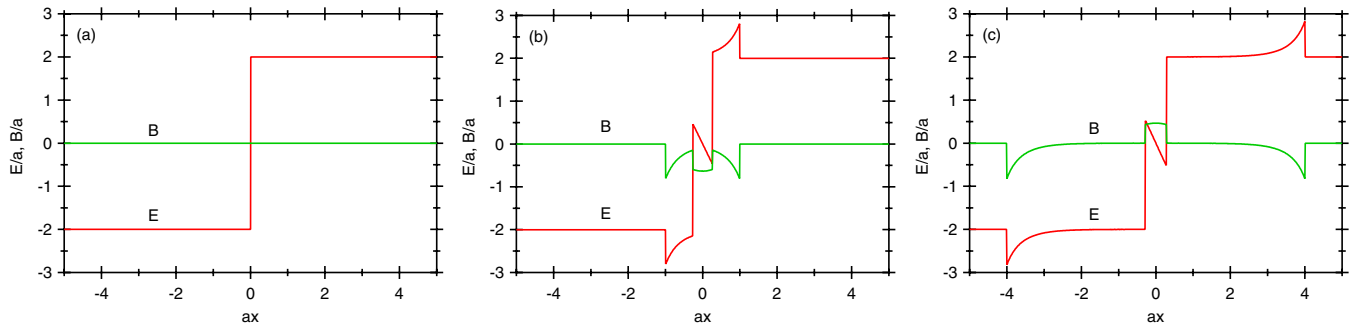


FIG. 5 (color online). Two particles. E/a and B/a versus ax . (a) $at = 0$. (b) $at = 1$. (c) $at = 4$.

$$\begin{aligned}\frac{du}{dt} &= -\gamma(E_r + \beta(2af_0 + B_r)), \\ \frac{df}{dt} &= -(a/\gamma)f^2 + (2af_0 + B_r + \beta E_r)f,\end{aligned}\quad (68)$$

where $\gamma \equiv w^0$, $\beta \equiv w^1/w^0$, $u \equiv \beta\gamma = w^1$,

$$B_r = -\epsilon(z)E_r = -af(t_r)\gamma(t_r)(1 + \epsilon(z)\beta(t_r)), \quad (69)$$

and t_r is defined such that $t_r = t - |z^1(t) + z^1(t_r)|$.

We numerically integrate the equations of motion (68) subject to the initial conditions $z^1(0) = 0$, $u(0) = 1$, $f(0) = 1$. In Fig. 4 we plot $z^1(t)$, $u(t)$, and $f(t)$, and in Fig. 5 we plot the E and B fields at times $at = 0, 1, 4$. We see that the particles oscillate relative to one another due to their mutual attraction. There is an initial transient period during which pulses of radiation are emitted by the particles, but after this transient period the particles do not radiate and the amplitude of the oscillations remains constant. From the flat spacetime point of view, the persistence of the oscillations is a consequence of the fact that the radiation reaction force vanishes, so there is no radiation damping. From the curved spacetime point of view, the

persistence of the oscillations follows from the fact that there is no physical gravitational radiation, and hence a system of gravitationally bound particles cannot dissipate energy.

VIII. CONCLUSION

We have considered the dynamics of point particle sources in a $(1 + 1)$ -dimensional model of general relativity. By reinterpreting the model as a scalar field theory in flat spacetime, we have formally solved the initial value problem for the system. In the model system the gravitational field does not support freely propagating radiation, but it does mediate forces among the particles and gives rise to nontrivial particle dynamics. We have shown that in the flat-spacetime interpretation of the model this property manifests itself in the fact that the radiation reaction force on a point particle vanishes, despite the fact that the particle can scatter and emit radiation. We have illustrated our results by presenting several example solutions, including a solution for two point particles interacting via their mutual gravitational attraction.

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